

Sub-Poissonian photon statistics in a strongly coupled single-qubit laser

M. Marthaler, Pei-Qing Jin, Juha Leppäkangas and Gerd Schön

Institut für Theoretische Festkörperphysik and DFG-Center for Functional Nanostructures (CFN), Karlsruher Institut für Technologie, D-76128 Karlsruhe, Germany

E-mail: mmartha@tfp.uni-karlsruhe.de

Abstract. We investigate qubit lasing in the strong coupling limit. The qubit is given by a Cooper-pair box, and population inversion is established by an additional third state, which can be addressed via quasiparticle tunneling. The coupling strength between oscillator and qubit is assumed to be much higher than the quasiparticle tunneling rate. We find that the photon number distribution is sub-Poissonian in this strong coupling limit.

The coupling of an oscillator to a superconducting qubit [1, 2], to a single-electron transistor [3, 4, 5], and directly to a tunnel junction [6] are promising possibilities to study quantum effects in electromagnetic or mechanical oscillators [7, 8]. On a principle level, all of those schemes can be used in multiple ways, for example for qubit read-out using the oscillator as detector [9], or the qubit can be seen as an artificial atom that can heat and cool the oscillator [10, 11, 12]. This has been realized in two systems. A flux qubit has been used to heat and cool an LC-oscillator [13, 14], while lasing has been realized using a Cooper-pair box and quasiparticle tunneling to address a third state [15, 16] to create population inversion. We will investigate the single-qubit laser in the limit of strong coupling [17]. This regime has an interesting lasing state with sub-Poissonian photon statistics. Our analysis extends previous studies in the strong coupling regime of the micro maser [18, 19] to a system where the atom is permanently coupled strongly to the cavity.

We assume that our qubit is given by a Cooper-pair box coupled via the island charge to the oscillator. The qubit eigenstates are a superposition of the charge states $|N = 0\rangle$ and $|N = 2\rangle$ and are given by

$$|\uparrow\rangle = \cos \frac{\phi}{2} |N = 2\rangle - \sin \frac{\phi}{2} |N = 0\rangle, \quad |\downarrow\rangle = \sin \frac{\phi}{2} |N = 2\rangle + \cos \frac{\phi}{2} |N = 0\rangle. \quad (1)$$

The rotation angle is given by $\tan \phi = E_J/4E_C\delta N_G$, where E_J is the Josephson energy of the junction, E_C is the charging energy of the island, and δN_G is the gate charge. The coherent time evolution of the system is determined by the extended Jaynes-Cummings Hamiltonian ($\hbar = 1$)

$$H = \frac{1}{2}\Delta E\sigma_z + g(\sigma_z \cos \phi + \sigma_x \sin \phi)(a^\dagger + a) + \omega a^\dagger a. \quad (2)$$

A third state $|N = 1\rangle$ is involved in the lasing cycle, but here it is only included on the level of the master equation. In the strong coupling limit the time evolution of the density matrix is

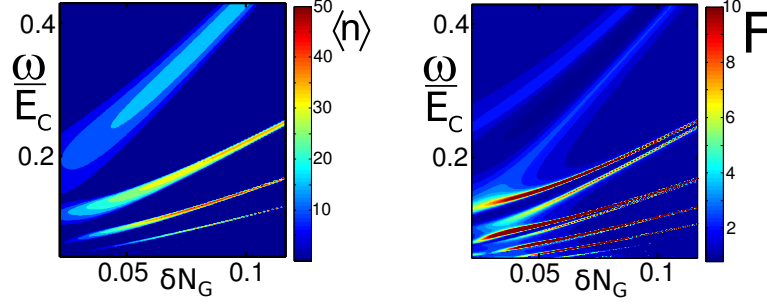


Figure 1. The average photon number $\langle n \rangle$, the Fano factor F as a function of the frequency ω and the gate charge δN_G . We see maxima in the photon number for the resonance condition $\Delta E = m\omega$, where $m = 1, 2, \dots$. At resonance, the Fano factor has a minimum and the current has a maximum. The parameters are $E_J/E_C = 0.18$, $\Delta/E_C = 2.2$, $eV/E_C = 7$, $g/E_C = 0.01$, $\kappa/(E_C/e^2 R_T) = 0.1$.

given in sufficient approximation by a simple balance equation

$$\dot{\rho}_i = \sum_j (\Gamma_{j,i} \rho_j - \Gamma_{i,j} \rho_i), \quad (3)$$

where $\rho_i = \langle i | \rho | i \rangle$, is the probability of the system to be in the state $|i\rangle$, $\Gamma_{j,i} = \Gamma_{j,i}^{\text{qp}} + \Gamma_{j,i}^{\text{diss}}$ is the transition rate from state $|j\rangle$ to state $|i\rangle$ as obtained from the Golden rule,

$$\Gamma_{j,i}^{\text{qp}} = |\langle i | \hat{T} | j \rangle|^2 I(E_{ji} + eV), \quad \Gamma_{j,i}^{\text{diss}} = \frac{\kappa}{\omega} \frac{E_{ji}}{1 - e^{-E_{ji}/k_B T}} |\langle i | x | j \rangle|^2, \quad (4)$$

where $E_{ji} = E_j - E_i$ is the energy difference between initial and final state. We assume linear coupling of the oscillator to reservoir, $x = a^\dagger + a$, with coupling strength κ/ω . The operator $\hat{T} = |1\rangle\langle 2| + |0\rangle\langle 1|$ decreases the charge of the island by one. The current through a superconducting junction with resistance R_T is given by

$$I(E) = \frac{e}{R_T} \int d\omega f(\omega) (1 - f(\omega + E)) N(\omega) N(\omega + E), \quad (5)$$

where $N(E)$ is the dimensionless superconducting density of states and $f(\omega)$ is the Fermi function at temperature T .

The crucial ingredient for any laser is the creation of population inversion. For the specific example we discuss here population inversion is created by quasiparticle tunneling. Without the coupling to the oscillator the quasiparticle tunneling rates are given by $\Gamma_{\uparrow,1}^{\text{qp}} \approx \Gamma_{1,\downarrow}^{\text{qp}} \propto \sin^2(\phi/2)$, and $\Gamma_{1,\downarrow}^{\text{qp}} \approx \Gamma_{\uparrow,1}^{\text{qp}} \propto \cos^2(\phi/2)$. For $\delta N_G > 0$ we get $\cos^2(\phi/2) > \sin^2(\phi/2)$. This means for the rates that $\Gamma_{1,\uparrow} > \Gamma_{\uparrow,1}$ and $\Gamma_{\downarrow,1} > \Gamma_{1,\downarrow}$. Therefore the system is most likely to be in the state $|\uparrow\rangle$. This creates population inversion in our qubit and can be used to generate lasing if the system is coupled to an oscillator. The energy for this process is provided by the transport voltage eV .

We can numerically calculate the eigenstates as a combination of the Fock states of the oscillator $|n\rangle$ and the eigenstates of the Cooper-pair box $|N=1\rangle, |\uparrow\rangle, |\downarrow\rangle$. Equation (3) can be solved numerically in the stationary case $\dot{\rho}_i = 0$. We are interested in the average excitation of the oscillator $\langle n \rangle = \langle a^\dagger a \rangle$ and the width of the distribution around this average, which defines the Fano factor $F = (\langle n^2 \rangle - \langle n \rangle^2) / \langle n \rangle$. Results for $T = 0$ can be seen in fig. (1). As expected we observe maximal excitation if the oscillator is at resonance with the

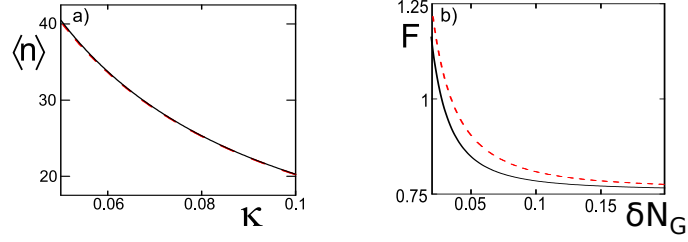


Figure 2. We compare analytical and numerical solutions for the average photon number $\langle n \rangle$ and the Fano factor F . Black line: analytical solution, red-dashed: numerical solution. a) $\langle n \rangle$ as a function of κ . The photon number decreases like κ^{-1} . For this plot we have chosen $\delta N_G = 0.1$. b) The Fano factor F as a function of δN_G . We see that the Fano factor becomes smaller than one for large δN_G . For this plot we have chosen $\kappa/(E_C/e^2 R) = 0.1$. For both plots we used the parameters $E_J/E_C = 0.18$, $\Delta/E_C = 2.2$, $eV/E_C = 7$, $g/E_C = 0.01$.

energy difference ΔE . The Fano factor at these positions becomes especially small, which shows us that the density distribution has a sharp peak around the average value of the photon number $\langle n \rangle$. Directly at resonance the Fano factor can even be smaller than one, which means that we have a sub-Poissonian distribution. One can also observe higher order resonances, $\Delta E = m\omega$, for $m = 1, 2, \dots$. Close to this resonances our master equation is valid for $g \left(g\sqrt{\langle n \rangle}/\omega \right)^{m-1} \gg \kappa, V/eR_T$, where m is the corresponding order of the resonance. Only if this condition is fulfilled the off-diagonal matrix elements of the density matrix can be neglected.

To understand better the emergence of sub-Poissonian photon statistics we analyze the system at the resonances. On resonance the eigenstates are given by the dressed states $|\pm, n\rangle = (|\uparrow\rangle|n\rangle \pm |\downarrow\rangle|n+m\rangle)/\sqrt{2}$. We have now two sets of states, the even states $|\pm, n\rangle$ and the odd states $|1, n\rangle = |N=1\rangle|n\rangle$. Quasiparticle tunneling leads to transitions between even and odd charges, this means it will cause transitions between the two sets. Oscillator dissipation does not change the charge of a state and therefore only causes transitions within each set. We can plug the eigenstates into the expression for the transition rates (4). The quasiparticle tunneling rates between the dressed states are given by the same rates as tunneling between the eigenstates of the qubit as long as $E_{ij} \ll eV$. Assuming that $\langle n \rangle$ is large we can approximate the oscillator dissipation rates as

$$\Gamma_{(1/\pm, n), (1/\pm, n-1)}^{\text{diss}} = (\bar{n} + 1)n\kappa, \quad \Gamma_{(1/\pm, n), (1/\pm, n+1)}^{\text{diss}} = \bar{n}(n + 1)\kappa, \quad (6)$$

where $\bar{n} = (e^{\omega/k_B T} - 1)^{-1}$. Using theses rates we can write the balance equation (3) as

$$\begin{aligned} \dot{\rho}_{\pm, n} &= \frac{1}{2}\Gamma_{1, \uparrow}^{\text{qp}}\rho_{1, n} + \frac{1}{2}\Gamma_{1, \downarrow}^{\text{qp}}\rho_{1, n+m} + (\bar{n} + 1)(n + 1)\kappa\rho_{\pm, n+1} + \kappa\bar{n}n\rho_{\pm, n-1} \\ &\quad - \left(\frac{1}{2}\Gamma_{\downarrow, 1}^{\text{qp}} + \frac{1}{2}\Gamma_{\uparrow, 1}^{\text{qp}} + (\bar{n} + 1)n\kappa + \bar{n}(n + 1)\kappa \right) \rho_{\pm, n}, \\ \dot{\rho}_{1, n} &= \sum_{\pm} \left(\frac{1}{2}\Gamma_{\uparrow, 1}^{\text{qp}}\rho_{\pm, n} + \frac{1}{2}\Gamma_{\downarrow, 1}^{\text{qp}}\rho_{\pm, n-m} \right) - \left(\frac{1}{2}\Gamma_{1, \uparrow}^{\text{qp}} + \frac{1}{2}\Gamma_{1, \downarrow}^{\text{qp}} \right) \rho_{1, n} \\ &\quad + (\bar{n} + 1)\kappa((n + 1)\rho_{1, n+1} - n\rho_{1, n}) + \bar{n}\kappa(n\rho_{1, n-1} - (n + 1)\rho_{1, n}). \end{aligned} \quad (7)$$

We can now find a solution for the average oscillator excitation $\langle n \rangle$. To do this we multiply eqs. (7) by n and sum over all n . We know that for sufficiently low dissipation the distribution is peaked around $\langle n \rangle \gg 1$, therefore we can neglect $\rho_{i, 0}$. In the stationary limit we get a set of

linear equations which can be solved for $n_i = \sum_n n \rho_{i,n}$ ($\{i \in +, -, 1\}$). The average number of photons is then given in good approximation by $\langle n \rangle = n_+ + n_- + n_1$. This yields

$$\langle n \rangle = \frac{(\Gamma_{\downarrow,1}^{\text{qp}} \Gamma_{1,\uparrow}^{\text{qp}} - \Gamma_{\uparrow,1}^{\text{qp}} \Gamma_{1,\downarrow}^{\text{qp}})m}{\kappa (\Gamma_{\downarrow,1}^{\text{qp}} + \Gamma_{\uparrow,1}^{\text{qp}} + 2(\Gamma_{1,\uparrow}^{\text{qp}} + \Gamma_{1,\downarrow}^{\text{qp}}))} + \bar{n} \quad (8)$$

We see that at resonance the oscillator excitations are inversely proportional to the dissipation rate κ and linear in the order of the resonance m . The equation we derived here is rather general and describes any strongly coupled three-level laser. We can now use the same method to calculate higher moments of the photon number n . If we multiply equations (7) with n^2 and sum over all n it is straight forward to derive an approximate equation for $\langle n^2 \rangle$. For large photon numbers ($\Gamma_{\uparrow,1}^{\text{qp}}, \Gamma_{1,\downarrow}^{\text{qp}} \gg \Gamma_{\downarrow,1}^{\text{qp}}, \Gamma_{1,\uparrow}^{\text{qp}}$) the Fano factor reduces to

$$F \approx \frac{1}{2} \left(1 + \frac{\Gamma_{\uparrow,1}^{\text{qp}^2} + 4\Gamma_{1,\downarrow}^{\text{qp}^2}}{(\Gamma_{\uparrow,1}^{\text{qp}} + 2\Gamma_{1,\downarrow}^{\text{qp}})^2} \right) + \bar{n}. \quad (9)$$

Here we see that at the first order resonance ($m = 0$), low temperatures $\bar{n} \approx 0$ and for strong coupling, the Fano factor can be smaller than one. In fig. 2a) we compare the analytical results with the numerics. We have chosen δN_G and κ such that $\langle n \rangle$ is large enough to fulfill all approximations we have made to derive equations (8) and (9). In this case the numerical and analytical results fit perfectly.

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